Problem of the Week

September 18, 2007

Code

	A	**B**	**C**	**D**	**E**	**F**	**G**	**H**	**I**	**J**	**K**	**L**	**M**	**N**	**O**	**P**	**Q**	**R**	**S**	**T**	**U**	**V**	**W**	**X**	**Y**	**Z**
A	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s	t	u	v	w	x	y	z
B	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s	t	u	v	w	x	y	z	a
C	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s	t	u	v	w	x	y	z	a	b
D	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s	t	u	v	w	x	y	z	a	b	c
E	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s	t	u	v	w	x	y	z	a	b	c	d
F	f	g	h	i	j	k	l	m	n	o	p	q	r	s	t	u	v	w	x	y	z	a	b	c	d	e
G	g	h	i	j	k	l	m	n	o	p	q	r	s	t	u	v	w	x	y	z	a	b	c	d	e	f
H	h	i	j	k	l	m	n	o	p	q	r	s	t	u	v	w	x	y	z	a	b	c	d	e	f	g
I	i	j	k	l	m	n	o	p	q	r	s	t	u	v	w	x	y	z	a	b	c	d	e	f	g	h
J	j	k	l	m	n	o	p	q	r	s	t	u	v	w	x	y	z	a	b	c	d	e	f	g	h	i
K	k	l	m	n	o	p	q	r	s	t	u	v	w	x	y	z	a	b	c	d	e	f	g	h	i	j
L	l	m	n	o	p	q	r	s	t	u	v	w	x	y	z	a	b	c	d	e	f	g	h	i	j	k
M	m	n	o	p	q	r	s	t	u	v	w	x	y	z	a	b	c	d	e	f	g	h	i	j	k	l
N	n	o	p	q	r	s	t	u	v	w	x	y	z	a	b	c	d	e	f	g	h	i	j	k	l	m
O	o	p	q	r	s	t	u	v	w	x	y	z	a	b	c	d	e	f	g	h	i	j	k	l	m	n
P	p	q	r	s	t	u	v	w	x	y	z	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o
Q	q	r	s	t	u	v	w	x	y	z	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p
R	r	s	t	u	v	w	x	y	z	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q
S	s	t	u	v	w	x	y	z	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r
T	t	u	v	w	x	y	z	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s
U	u	v	w	x	y	z	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s	t
V	v	w	x	y	z	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s	t	u
W	w	x	y	z	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s	t	u	v
X	x	y	z	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s	t	u	v	w
Y	y	z	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s	t	u	v	w	x
Z	z	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s	t	u	v	w	x	y

You have managed to get hold of the above and are told that it is the template for a code. You have reason to believe that it uses the key word “Pierrepont” but you don’t know how it works. Decode the following:

IEEJ SVXZYBV IRU KLT GYBIPC KFZTG
QBS OCIV ECR TBBJPV ZR IVR PPJI

Problem of the Week

September 25, 2007

Code II

P	**I**	**E**	**R**	**O**
N	**T**	A	B	C
D	F	G	H	K
L	M	Q	S	U
V	W	X	Y	Z

As the old code has been cracked, a new one has appeared. Once again you have the template for a code, once again using Pierrepont as a key word. You work out that "Pierrepont" has been written out in the grid (removing all duplicate letters) and that all the other letters, except for J, follow in order. (that isn't part of the code…since 26 is not a perfect square, any J's in the original message are replaced by I).

Fortunately, you also got hold of one example. The phrase "Autumn in New England" is rendered as:

CQ CM LT PT VA IX PA DQ BT GV.

Can you decode the following?

CI AR PO CP AW CI AR BF BA RM BF AX QO MB EP AV

Problem of the Week

September 25, 2007

Snowball

What rule was followed in writing the following sentence?

I am not sure about poetry written linearly, following impersonal mathematics mechanically.

Can you construct a sentence according to the same rule?

P				L				F				W			
	R		B		E		O		T		E		E		K
		O				M				H				E	

October 2, 2007

Bouncing Ball Codes

Frustrated by our school's persistent ability to obtain key words and to crack codes, the unknown message senders have decided to explore codes with no key word. The heading gives you an example: "Problem of the Week" would be encoded as:

PLFWRBEOTEEKOMHE

Of course, there are many variants of this basic principle...the above is just an example. Here are some messages for you to decode:

(first, using the same code)

IUNTIEAREARNIGUOSMLIESOCDSMNOFPDFO

(next, using a very similar code)

NBENEOAONEIKDRDTUIDHURCSOBTTTTPAOEUIOHD

(and, finally, using a variant)

NEEEEINSHWTOEKWPMNGTCIUNXTIRSVCHOLRTHLONOO ODLNWELBOLDUERSERBLTIEGSEO

Problem of the Week

October 2, 2007

Names

A family has four children:

- Mary grew up to be a soldier.
- Dora became a racecar driver.
- Dan spends most of his time joining things.
- The fourth child became a mind reader.

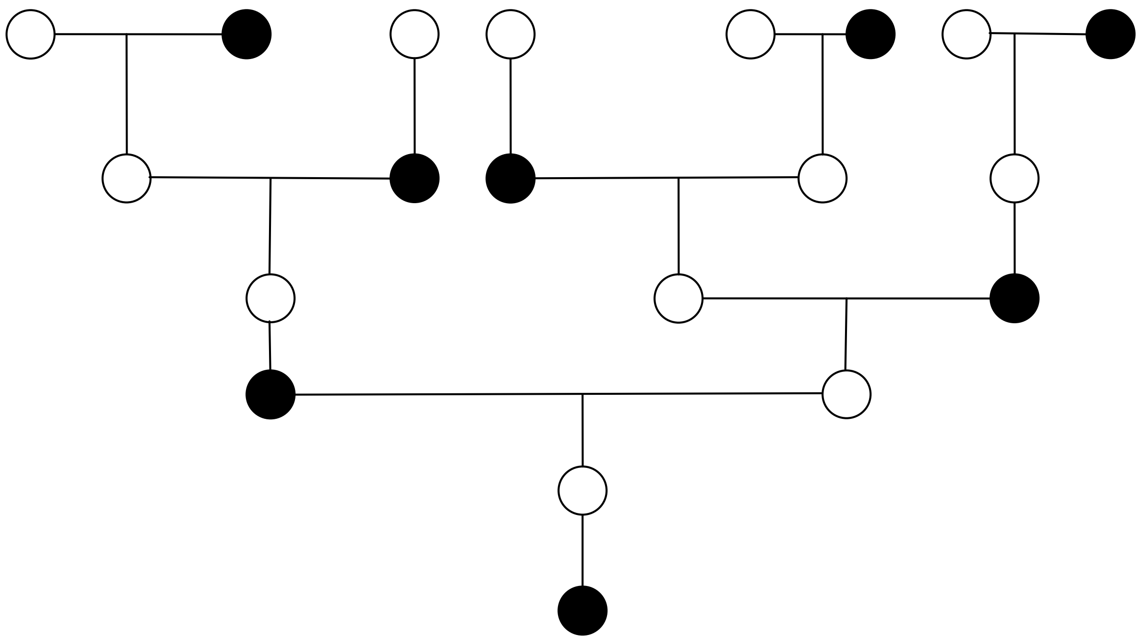
What was the name of the fourth child?

Problem of the Week

October 9, 2007

Bee Trees

Honey bees reproduce in a curious way. Male bees are born asexually; that is, they have only one parent (a female). Females are born of two parents (a male and a female). Shown below are six levels of the family tree of a typical male bee:

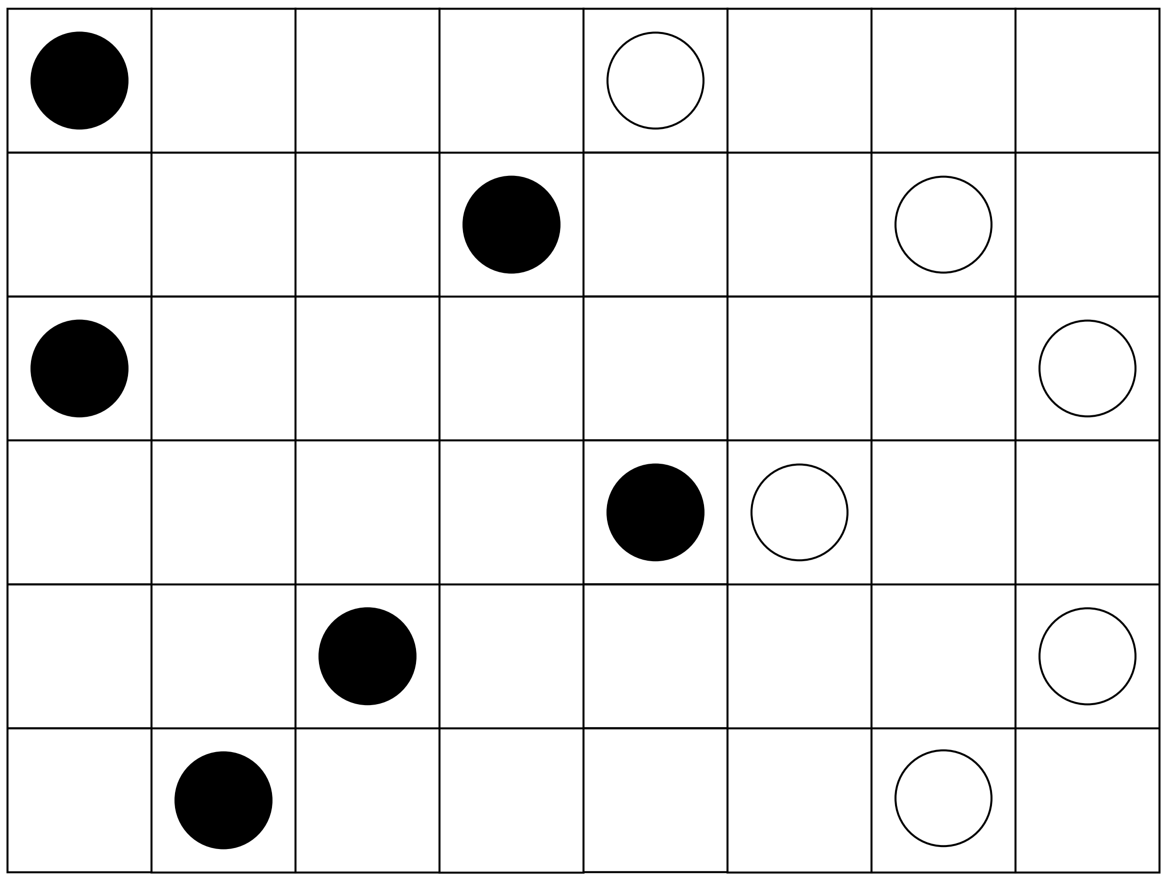


Here black circles denote males and white circles denote females. You can see, for example, that the boy bee at the bottom has 8 ancestors five generations back (you, by way of contrast, have 32). You can also see that the tree for a girl bee looks exactly the same, just one level off (so the girl's parents are the boy's grandparents, and so on).

How many ancestors has a boy got six generations back? Ten generations back? Twenty? How many of those were female?

Problem of the Week October 16, 2007

Northcott's Game

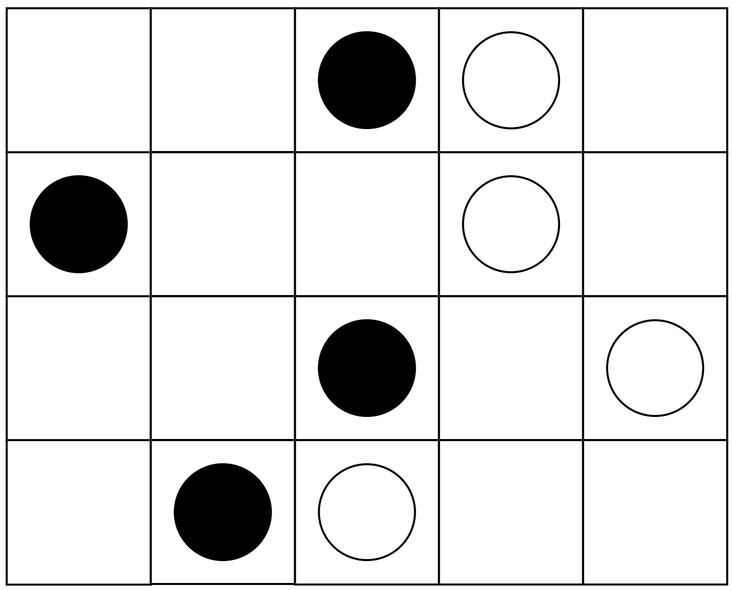


This game can be played on a grid of any size and with many different starting positions (it is required only that each row have exactly two disks, one black and one white, with the black disk to the left of the white disk). One player controls the black disks, the other controls the white. Taking turns, the players must move one of their disks. Their disk can slide in its row any number of squares in either direction, but it can not jump over the other disk. A player loses if all of their disks are trapped by the enemy disks on the side. Starting with the above position, and assuming that neither player makes any mistakes, which player should win? Note: it isn't even clear that the game has to end. If both players cooperate the game can certainly go on forever.

Problem of the Week

October 16, 2007

Northcott's Game (simpler version)



This game can be played on a grid of any size and with many different starting positions (it is required only that each row have exactly two disks, one black and one white, with the black disk to the left of the white disk). One player controls the black disks, the other controls the white. Taking turns, the players must move one of their disks. Their disk can slide in its row any number of squares in either direction, but it can not jump over the other disk. A player loses if all of their disks are trapped by the enemy disks on the side. Starting with the above position, and assuming that neither player makes any mistakes, which player should win? Note: it isn't even clear that the game has to end. If both players cooperate the game can certainly go on forever.

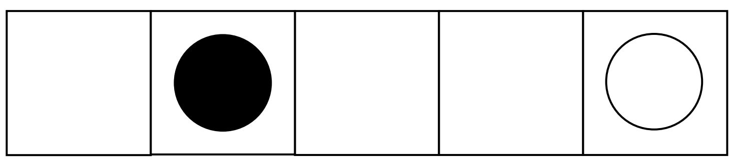
Problem of the Week

October 23, 2007

Northcott's Game II

Charlie has a conjecture concerning the game…so far as I know, no student has been able to support or reject it. He says this: "Count the number of spaces between the checkers. If the total is Odd, then Player I should be able to win."

Note: it is NOT true that an Even total always predicts a Win for Player II. Here is a counterexample:



Can you persuade yourself that Player I should be able to win in this situation?

Maybe you think this is a bad example, as there is only one row. Can you come up with a multi-row example with an Even number of spaces in which Player I can still force a win?

And what about Charlie's Conjecture? Can anyone find a counterexample? Can anyone explain why it is true?

Problem of the Week

October 23, 2007

Missing

Can you work out what is unusual about this paragraph? It is an illustration of a typical lipogram. All words in it look natural, I think, and it contains no truly non-standard or curious grammar. But it actually is odd, surprisingly abnormal, profoundly outlandish. Writing it, constructing it sharply, was hard work…not fast at all. Nor was it rapid, nor quick, nor any synonym of swift. In particular, nothing about it was hasty. Arranging this random looking mix of words took fully half an hour. You ought to study it! I think you will grow fond of it, though I can not say this with conviction (do you distrust voids and vacuums, do you find gaps awkward and discomforting? or do such things amusingly occupy your mind?). Having first thought it through, you might try making such a paragraph on your own. Working in pairs or finding a coach sounds fair…why not? Good luck with it. May it bring you much joy! (Hint: using a dictionary to look up "lipogram" is a lazy path to a solution, so don't do it too soon and if you must, in cowardly fashion, look it up, don't impart this information to anybody. Okay, I must admit that this was a caution, and not a significant hint.)

Problem of the Week

October 30, 2007

The Charitable Stranger

An elderly fellow was dividing his gold coins between his three children, none of whom were terribly good at math. To the Eldest, he left 3/8 of the gold. To the Middle Child, he left 1/3 and to the Youngest, 2/7. " If there are any gold pieces left over", said the man, "share them equally, unless Providence in It's infinite wisdom should send you a Charitable Stranger, in which case give them to him." No sooner had he made his intentions clear then the poor man died.

Alas, the children found that the number of gold pieces was not divisible by 8, nor by 3, nor by 7 and they fell to quarrelling. At this point a Stranger (who was not at all Charitable by nature but who was good at math) was drawn by the noise and asked for an explanation. They explained the terms of the will to the stranger who promptly said "Ah! Well, happily I have a gold piece in my pocket which, in the interests of harmony, I am happy to give to you." Accepting the coin from the stranger they then found that the increased estate was evenly divisible by all three numbers (8, 3, and 7) and they carved up the estate accordingly. On finishing, they found that they had 30 pieces of gold left over which, of course, they gave to the (supposedly) Charitable Stranger, who walked off considerably wealthier than he once was.

How many coins were in the estate left by the old man?

If you were one of the children, how would you have divided the coins? Is your method strictly fair (according to the rules the old man left, that is)?

Problem of the Week November 5, 2007

Multiple Choice

1 How many questions on this test have the answer “a”?
  a. 0
  b. 1
  c. 2
  d. 3
  e. 4

2 How many questions on this test have the answer “b”?
  a. 0
  b. 1
  c. 2
  d. 3
  e. 4

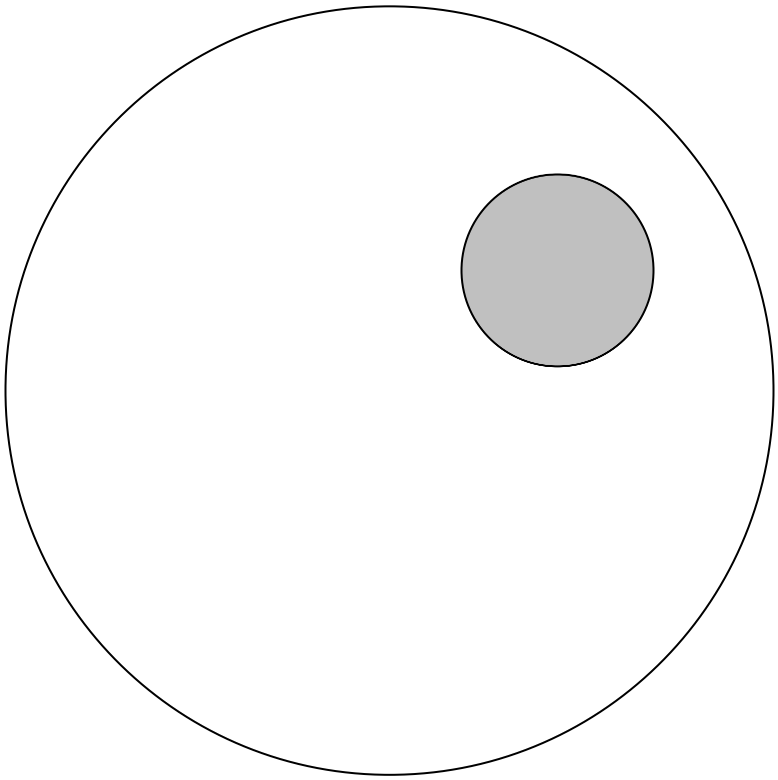
3 How many questions on this test have the answer “c”?
  a. 0
  b. 1
  c. 2
  d. 3
  e. 4

4 How many questions on this test have the answer “d”?
  a. 0
  b. 1
  c. 2
  d. 3
  e. 4

5 How many questions on this test have the answer “e”?
  a. 0
  b. 1
  c. 2
  d. 3
  e. 4

Problem of the Week

November 13, 2007



You have generously baked a cake from which someone has unkindly removed a circular piece, as shown (somewhat inartistically).

You need to divide the cake in two. Can you cut the cake fairly with one slice?

That is, can you cut the cake with one slice in such a way that the two pieces have equal area?

Problem of the Week November 13, 2007 Multiple Choice II

1. All of the following are true EXCEPT
   a. The answer to this question is not A
   b. The answer to question #2 is B.
   c. The only questions with vowels for answers are #4, #8, and #10.
   d. This is the only question with the answer D.
   e. Questions #8 and #10 have the same answer.

2. What is the answer to this question?
   a. A b. B c. C d. D e. E

3. The most frequent answer on this test is:
   a. B b. C c. D d. A e. E

4. The only answer on this test which doesn't occur more than once is:
   a. A b. B c. C d. D e. E

5. The first answer on this test to appear twice is:
   a. A b. B c. C d. D e. E

6. This is the last question on this test with the answer:
   a. A b. B c. C d. D e. E

7. The only other question on the test with the same answer as this one is:
   a. #1 b. #2 c. #3 d. #4 e. #5

8. All the odd numbered questions on this test have one of the two answers:
   a. A or B b. A or E c. D or E d. C or E e. C or D

9. The answer to question #7 is:
   a. D b. A c. E d. C e. B

10. The number I am thinking of is:
    a. 64,507 b. 31 c.102.5 d. 1 e. 31,212

Problem of the Week

November, 27, 2007

Place Value

In a strange act of rebellion, students at a certain school throw out our usual way of writing numbers. They keep the idea of place value, but they refuse to use what we'd call base 10. Unable to agree on a substitute, they split into two rival camps. Interviewing two of the kids you hear:

From Amy: "My group uses base 10. Of the 144 kids in the school, 60 of them are in my group."

From Bill: "My group uses base 10. Of the 84 kids in the school, 44 of them are in my group."

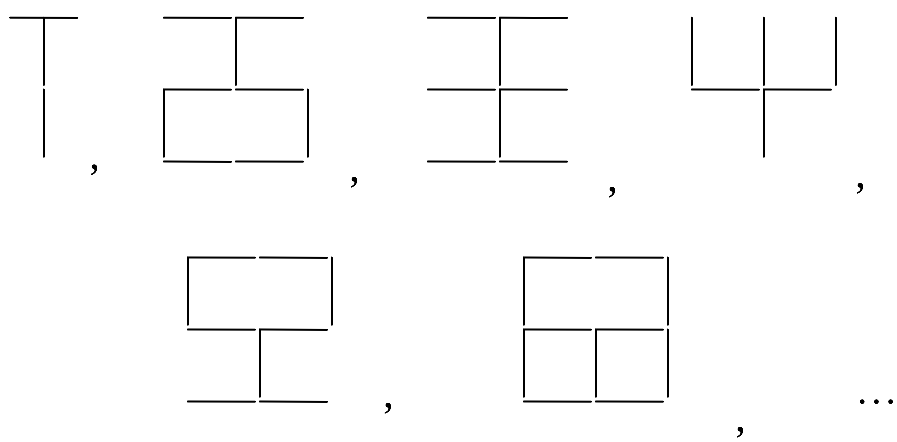
How many kids are in the school?

Problem of the Week

November 26, 2007

A Classic

What’s next in the sequence?



Problem of the Week

December 4, 2007

A Very Lazy Fellow

Once there was a Very Lazy Fellow who loudly complained that there was no way for him to make money without working. No sooner had he said this than the Devil appeared and made him a deal: "You can make money for nothing!", said the Devil. "Just say the word 'Double!' out loud and all the money you have in your pocket will be doubled. All I ask in return is that you pay me $100 whenever you do it."

This sounded very good to the Lazy Fellow, who promptly shouted "Double!". It worked, of course, and the Fellow happily handed the Devil $100. He said it again and again paid the price. Alas, when he said it a third time, he found that he wound up with exactly $100 and had to give all of it to the Devil, who laughed and vanished.

How much money did the Very Lazy Fellow start with?

Problem of the Week

December 4, 2007

Place Value II

As is the nature of rebellions, the passage of time has changed everything. Many more students have joined and a new dominant faction has emerged. You interview Carmen, the leader of that group, who tells you “we now have 280 students in the rebellion! 7/8 of them are with me and the other 59 will join soon enough.”

How many students are in the rebellion?

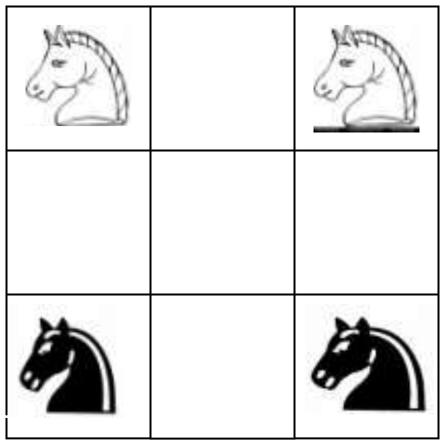
Hints:

- this can be solved algebraically, but you will encounter quadratic relations. If you aren’t used to those you might find it easier to proceed by trial and error.
- I fear it will take you a while to sort out long division in other bases. Multiplication is a lot easier. (a question: why is long division in other bases so very difficult?)

Problem of the Week

December 11, 2007

Guarini's Problem



4 knights are arranged on a 3 x 3 board, as shown. Your goal is to exchange them so that the two white knights currently in the top corners occupy the bottom corners, and the two black knights in the bottom corners occupy the top corners. Each knight moves in the usual L-shaped way, but there are no captures and the pieces may move in any order you like (for example, you might move a single piece several times in a row). Of course, each small square may only hold one knight at a time.

History: this is one of the oldest recorded puzzles, first appearing in print in 1512. Certainly it is the oldest known chess puzzle. If it is too easy for you, you might try to exchange 3 knights. Can it be done on such a small board? How much bigger does the board have to be?

Problem of the Week

January 8, 2008

Word + x

Here is an intriguing idea for a code, or for a poem, depending on how you look at it, and whether your intent is to confuse or to entertain. Take some existing text, such as a paragraph which explains the workings of the cipher. The passage you use must be long enough, and must contain a satisfying mixture of words (some routine and some extraordinary). The various parties then settle on some shift, a smallish sort of number, something between negative ten and ten, just to establish some norms. Someone wishing to encode a sentence would locate each of the words of the message in the pre-selected writing sample, and then write the word which precedes (or follows) it by the agreed upon value. Thus, if the word were, say "existing" and the negotiated numeral was positive five, I might find it in the second sentence…looking ahead five words, I would then be inspired to transmit "paragraph". There are, of course, severe limitations to this technique! First of all, each and every word of your message is required to be present in the sample. Not good at all. Secondly, a given word may appear in the template, and therefore in the cipher, many times…thereby compelling a potential decoder to try different combinations in an attempt to discover the true meaning. Those who seek to decode these messages must be able to cope with what will appear to be too many instances of "a" and far too many options in general. All much harder than the sort of codes we have been considering. And, to make matters considerably worse, you students may find yourselves faced with the prospect of untangling a seemingly meaningless jumble, having been given the special paragraph, but lacking the shift. You might be forced to work with codes like this:

Considerably true transmit messages and who severe to transmit of appear the sample to! Was passage be general might intent depending.

Problem of the Week

January 15, 2008

Natural Units

How many seconds tall are you?   How much do you weigh in meters?

Discussion: All units are arbitrary, but you may not understand just how arbitrary they are. In the system of *natural units* one sets several familiar physical quantities to 1, thereby simplifying many calculations. For example, the speed of light is typically set to 1. This answers the first of my two questions! The speed of light is 299,792,458 meters/second …if this is 1 then it must be that 299,792,458 meters = 1 second . I am about 2 meters tall…so my height in seconds is:

$$\frac{2}{299{,}792{,}458} = .000000006713 \ \textit{seconds}$$

As a slightly different way of putting it, this is how much time it takes light to go from my head to my feet. (not much time, really)

Your job is to answer the second question (your weight in meters). To do it you will need to set one other physical constant to 1…we'll use Newton's Gravitational Constant which governs gravity. In conventional (i.e. metric) units the Gravitational constant is given by:

$$G = 6.67428 * 10^{-11} \frac{meters^3}{(kilos)*(\sec)^2}$$

(it might help to know that 100 pounds is about 45.36 kilos).

Problem of the Week

January 22, 2008

The Game of Risk

In the game of Risk there are, among other things, Attackers and Defenders. Battles are fought with dice, happily, but the odds are not always so easy to sort out. Here are some basic probability questions which arise from the game:

One on One Attack: Each player rolls a single die and scores the value shown. The player with the higher score wins, with all ties going to the Defender. What are the odds that the Defender survives a One on One Attack?

Two on One Attack: The Defender rolls a single die and scores the value shown. The Attacker rolls a pair of dice and scores the *maximum* shown. Again, the player with the higher score wins, with all ties going to the Defender. What are the odds that the Defender survives a Two on One Attack?

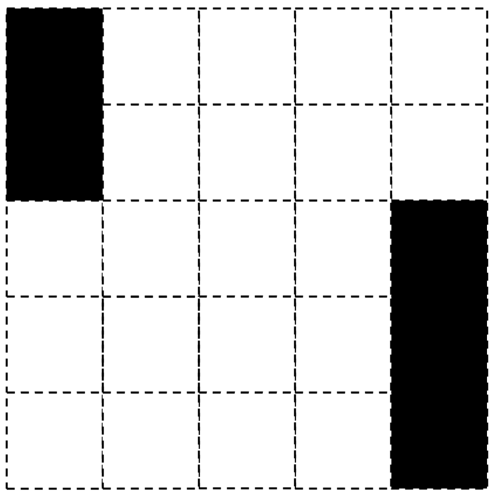
(To be clear, in each case the question concerns a single attack, with a single roll of however many dice. In the game itself, multiple attacks are common and generate their own probability questions)

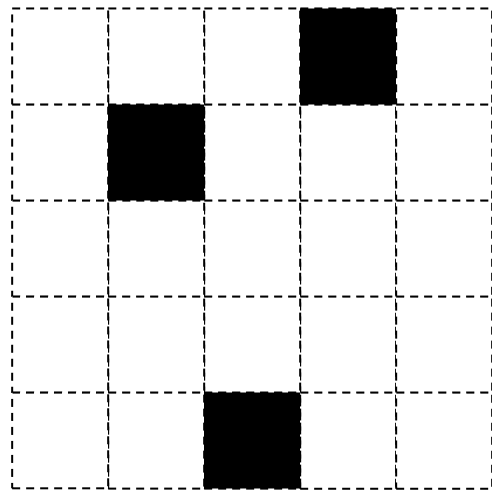
Note: the second question is considerably harder than the first.

Problem of the Week

January 29, 2008

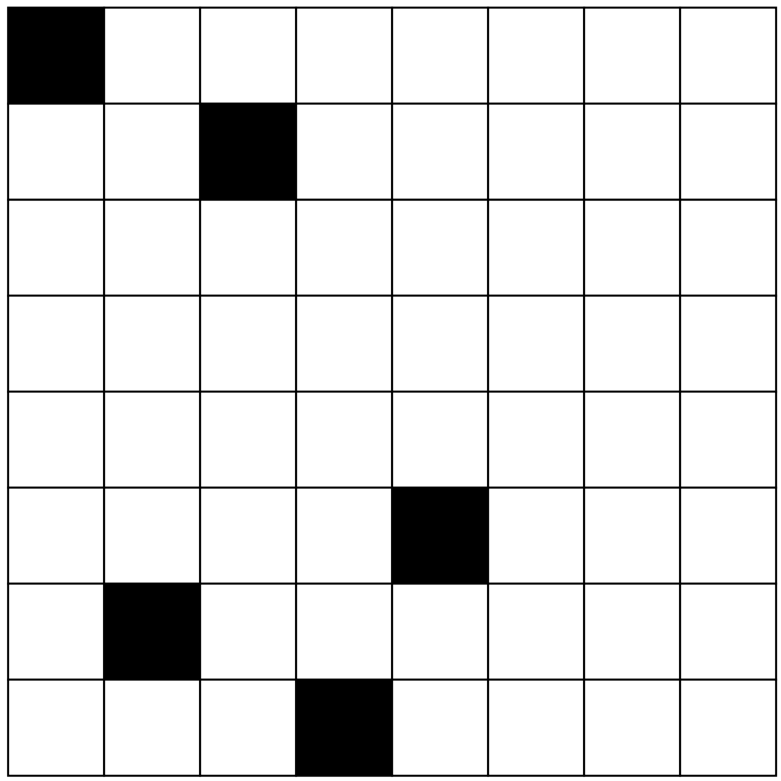
A Simple Game





In each of the above, your job is to draw a path which passes through each of the unfilled squares. Paths move vertically or horizontally, never diagonally. Your path can not cross itself, nor can it change direction except when needed to get around a filled square or to avoid crossing a previously drawn portion of the path. When the path is forced to change direction, you may choose whichever available direction you like (up, down, left, or right). You may start in any unfilled square you like.

Grid Game…a harder example



Question: can you draw a reasonably simple example for which no solution exists? You have to play fair…the unfilled squares have to be all connected, for example.

Problem of the Week

February 5, 2008

When Machines Go Bad

Here's a very simple operation. Take a number x between 0 and 1. Multiply your number by 2…if the result is still less than 1, that's your answer. If your result is 1 or more, your answer is 2x – 1. Repeat until a pattern emerges. If, for example, we had chosen 1/5 to start, we'd get:

$$\frac{1}{5} \to \frac{2}{5} \to \frac{4}{5} \to \frac{3}{5} \to \frac{1}{5}$$

As you see, this sequence begins to repeat itself.

First, try this by hand: Starting with a different fraction, does the sequence always repeat from the beginning? Does it start to repeat at some point? The number 1 is left unchanged by this operation, as is the number 0…are there any other such numbers? Which fractions (if any) eventually go to 0? Which fractions (if any) eventually go to 1?

Having worked this out by hand in a few examples, try it on a calculator. Try 1/5, try 1/9. Does it work? Go out reasonably far (depending on the calculator you might have to go 50 stages or so before serious problems appear).

What's going on here? The calculation is easy enough…you can do it by hand in seconds. Why can't the machine do it?

Problem of the Week

February 12, 2008

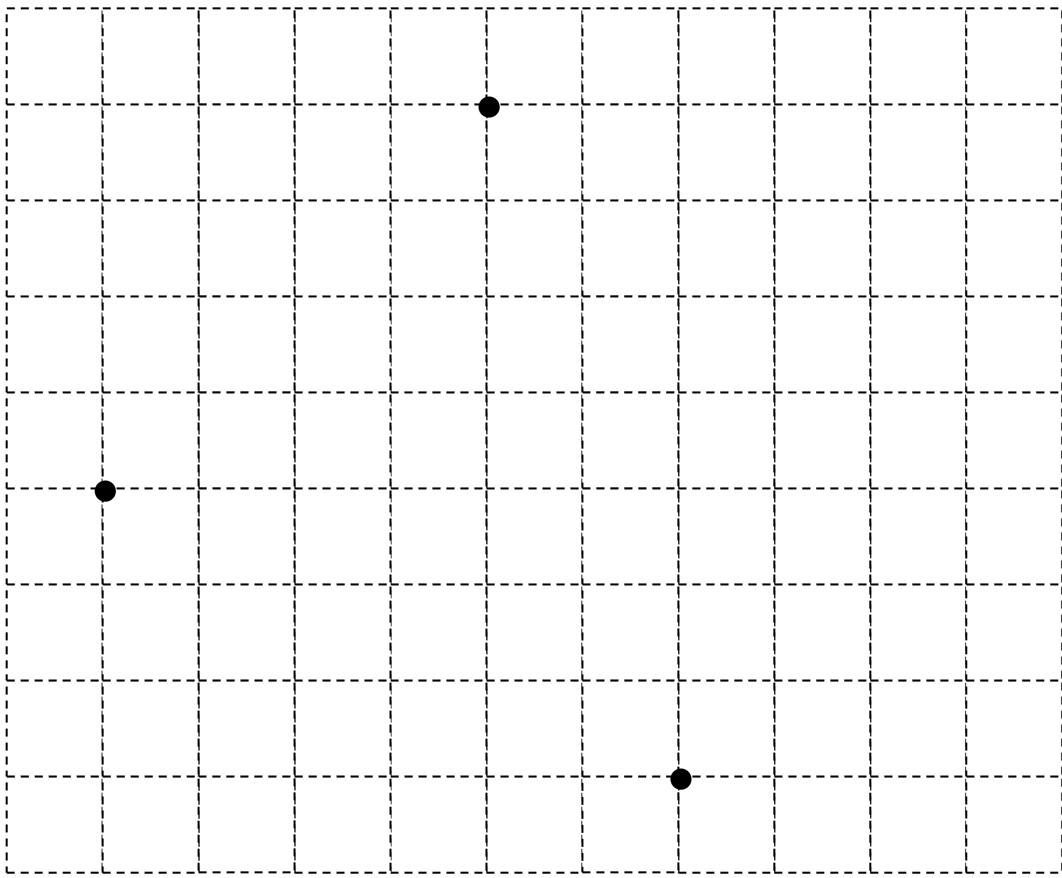
Breaking Bread

A hungry fellow walking in the woods comes across two travelers, one of whom has three loaves of bread and the other, five. The loaves are all identical and the three of them share the bread equally. The hungry fellow contributes no bread and pays $8 for his share…how should the other two distribute the money?

Problem of the Week

February 26, 2008

Placement



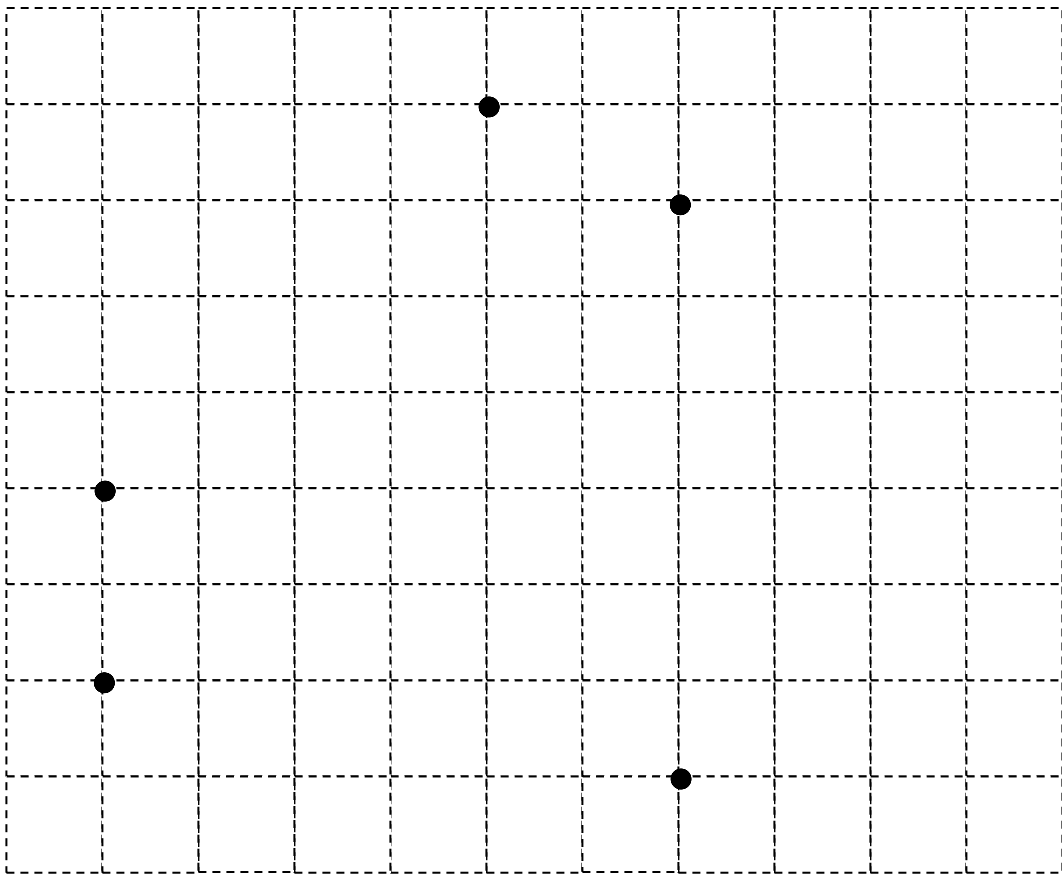
Shown above is a grid, indicating the roads in a city. Three newsstands are positioned at intersections, as shown. The town newspaper would like to drop the daily shipment at a single intersection; the papers would then be taken by hand to each of the three newsstands. Of course, the papers can only be carried along the streets…diagonals are not allowed. Assuming that the goal is to make the TOTAL distance over which the papers must be carried as small as possible, where should the newspapers be dropped?

Note: trial and error will work, of course, but can you come up with a general method of solution?

Problem of the Week

February 26, 2008

Placement (variant)



Shown above is a grid, indicating the roads in a city. Five newsstands are positioned at intersections, as shown. The town newspaper would like to drop the daily shipment at a single intersection; the papers would then be taken by hand to each of the newsstands. Of course, the papers can only be carried along the streets…diagonals are not allowed. Assuming that the goal is to make the TOTAL distance over which the papers must be carried as small as possible, where should the newspapers be dropped?

Note: trial and error will work, of course, but can you come up with a general method of solution? Your method should work for any ODD number of newsstands. What goes wrong with even numbers?

Problem of the Week

March 4, 2008

Babel

17 students are chosen from our school. It is known that, between them, they study Latin, Greek, and Chinese. Of course, not all students study all three. It is observed that any given pair has a single preferred language between them. That is, Students A and B speak only Greek to each other, even though both happen to know Chinese. Students B and C speak Latin with each other, and so on.

Show that we can always find 3 students in this group who like to speak one language between them.

Problem of the Week

March 11, 2008

Big Number

11, 047, 694, 236, 668, 359, 048, 016, 134, 593, 027

The number above happens to be the 17^{th} power of an integer. Using this fact, but without using any sort of calculator, can you find that integer?

Hints:

1. first work out how many digits the integer has.
2. is the integer divisible by 5?
3. is the integer divisible by 3?
4. (harder) what is the last digit of the answer?
5. when playing games with powers of integers it is often helpful to use the approximation

$$2^{10} \approx 10^3$$

(the left hand is exactly 1,024)

Problem of the Week

March 11, 2008

Perfect Squares

A *perfect square* is a whole number, like 16, that is the square of another whole number, as 16 is the square of 4 (16 = 4*4).

Can you quickly determine whether or not the number

456, 786, 466, 555, 169, 961, 042

is a perfect square?

Note: don't try to find the square root! There is an easier way to decide whether or not this particular number is a perfect square.

Full Disclosure: This problem was suggested by my brother.

Problem of the Week

March 18, 2008

Palindromes I

Note: A *Car Talk* problem, proposed by Abraham.

I was driving on the highway recently and I happened to notice my odometer. Like most odometers nowadays, it shows six digits, in whole miles only--no tenths of a mile. So, if my car had 300,000 miles, for example, I'd see 3-0-0-0-0-0. And that's all. Until I drove another mile, at which point it would read 3-0-0-0-0-1.

Now, what I saw that day was very interesting. I noticed that the last 4 digits were palindromic, that is they read the same forwards as backwards. For example, '5-4-4-5' is a palindrome. So, my odometer could have read 3-1-5-4-4-5, with those last four digits, starting with the units, then the tens, then the hundreds, and finally the thousands, being the palindrome.

One mile later, the last 5 numbers were palindromic. For example, it could have read 3-6-5-4-5-6.

One mile after that, the middle 4 out of 6 numbers were palindromic. So, the first and last numbers weren't involved in the palindrome, but the middle 4 were palindromic.

One mile later, all 6 were palindromic! For example, 2-1-3-3-1-2.

What did I see on the odometer when I first looked?

Problem of the Week

March 18, 2008

Palindromes II

An integer is a *palindrome* if, like 888, or 167,761 , it reads the same backwards as forwards.

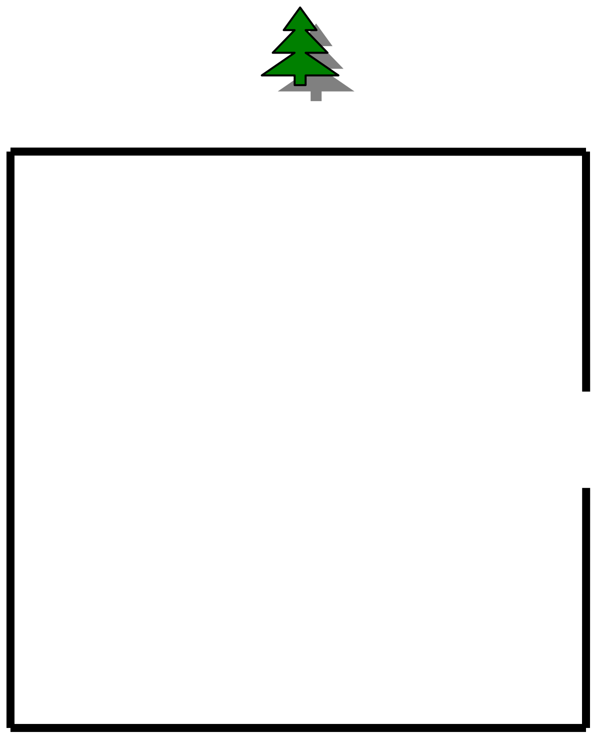
How many palindromes can you find which are prime and which have evenly many digits?

Note: there are many examples with an odd number of digits. Can you find some? To the best of my knowledge, it is not known if there are infinitely many.

Problem of the Week

March 25, 2008

Similar Triangles I



Imagine a square house, 60 feet on a side, with a door facing East (the door is, as shown, in the middle of the Eastern side).

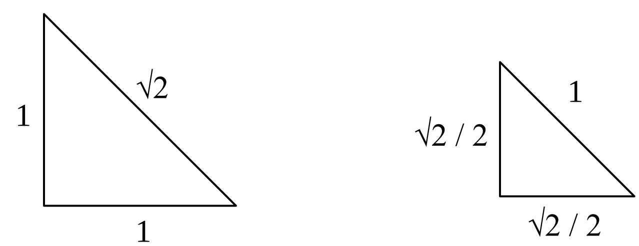
A Lovely Tree is located 15 feet from the middle of the Northern side. If you were to leave the house by the East door, and trusting that you continue to walk due East, how far would you have to walk before you could turn and see the Lovely Tree?

Problem of the Week

March 25, 2008

Similar Triangles II

It's well known that specifying the three sides of a triangle is enough to determine the triangle. That is, any two triangles with the same three sides must be copies of each other. It is clearly not true that knowing the three angles is enough, as one triangle might be a scaled version of the other. Nor is it enough to know three angles and one side, *unless you are told which side it is.* If, for example, we are told that the triangle has angles {45, 45, 90} and that one of the sides has length 1, then the triangle might be either of these two:



Question: Is it enough to know all three angles and <u>two</u> sides?

Just to be clear: you aren't told where the given sides are in relation to the angles, you just get the three angles and two of the sides. That's it. To emphasize; in the example above, two sides would obviously settle the matter.

Similar triangles (solution)

No, it is not enough. (though it is usually enough)

To see this, let's construct a counterexample. Since we know the angles, the pair of triangles in our counterexample must be similar. That means that the sides are proportional. Suppose the sides of the first triangle are {A, B, C} with $A \leq B \leq C$. The other triangle must then have sides {xA, xB, xC} for some constant x, which I might as well suppose to be greater than 1 (if not, just reverse the role of the two triangles). Clearly xC is larger than any of A, B, or C (and A is smaller than xA, xB, or xC) so the only hope is that $B = xA$ and $C = xB$. It follows that my first triangle must have had the form $\{A, xA, x^2 A\}$. We might as well suppose that A was 1, so our triangle is $\{1, x, x^2\}$ with $x > 1$.

Is this possible? Well, yes…but it isn't easy. You couldn't, for instance, take x to be 2, because {1, 2, 4} can't be the sides of a triangle (Why not?). I need these three numbers to be the possible sides of a triangle, so I need the triangle inequality to be satisfied. Thus we require that $x^2 < 1 + x$. But a little algebra then shows that x must be less than the Golden Ratio; we have:

$$1 < x < \frac{1+\sqrt{5}}{2} \approx 1.618$$

that narrow strip gives all possible counterexamples!

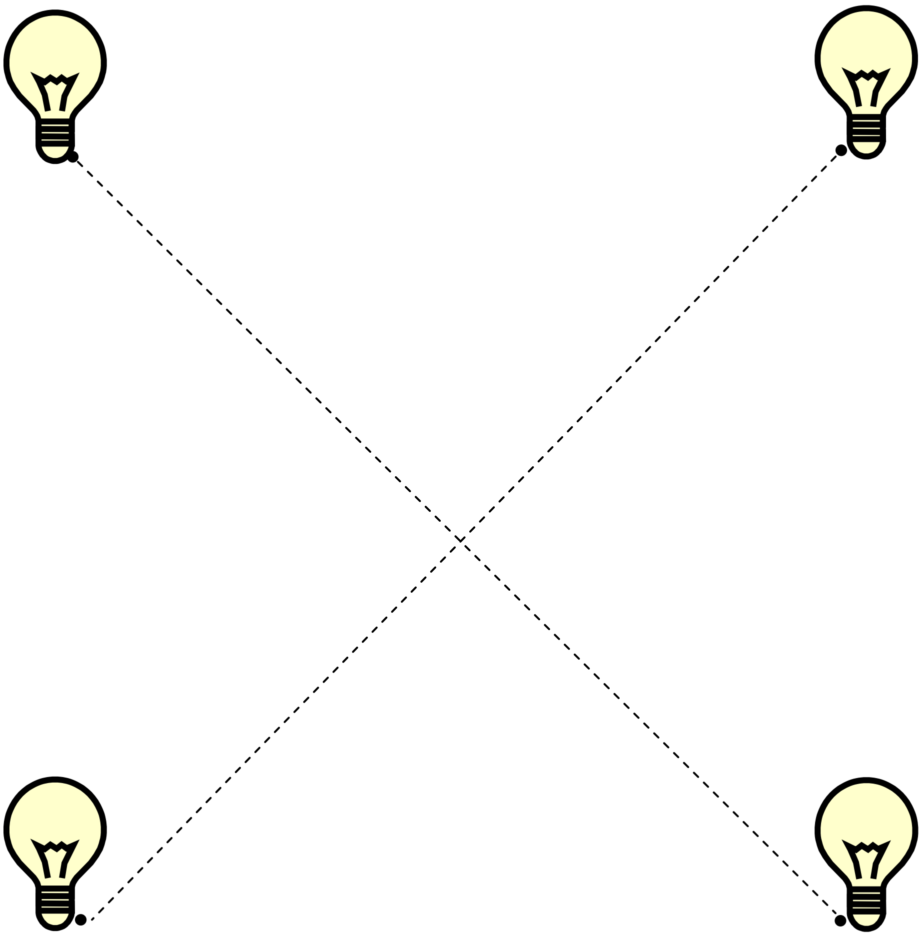
As a concrete counterexample, we might have the triangles with sides {1, 1.5, 2.25} and {1.5, 2.25, 3.375}.

It is possible to construct a counterexample out of right triangles. Requiring that $\{1, x, x^2\}$ be a right triangle ends up forcing x to be the square root of the Golden Ratio, or approximately 1.27 .

Problem of the Week

April, 1 2008

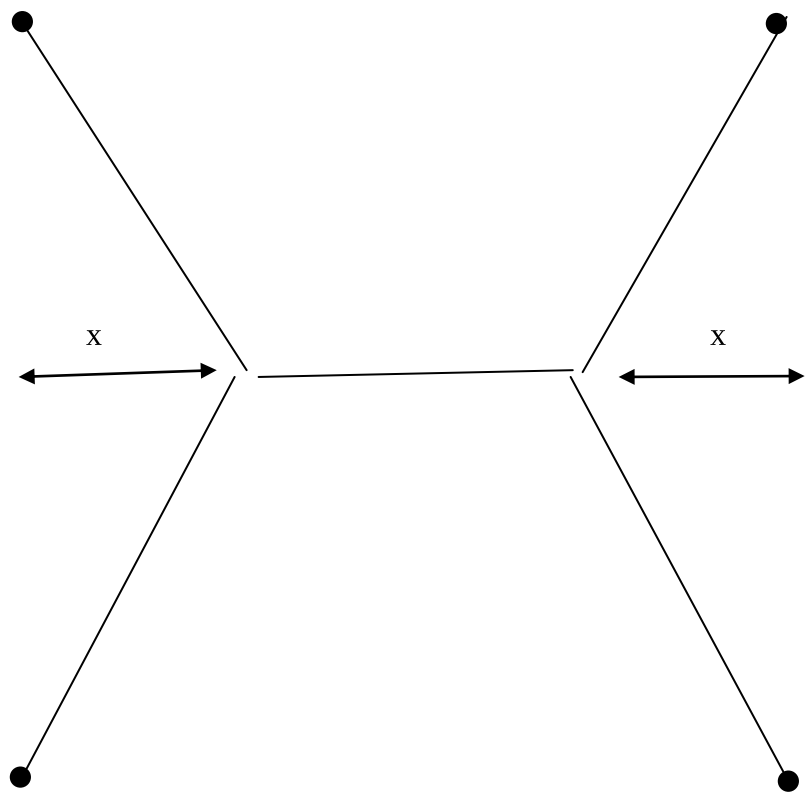
Wiring



Four light bulbs are arranged as you see them, on the corners of a four foot by four foot square. Your goal is to wire them together so that they can be powered efficiently. You had planned to do it in the crossed fashion, as shown. Alas, that requires a little more than 11.3 feet of wire (why?) and it turns out that you only have a bit less than 11 feet. How can you achieve your goal? Note: as shown, the square really is 4 inches by 4 inches (as measured from the 4 dots), so you can actually do this using 11 inches of string.

Note: the optimal answer will not have any curves in it. Why not?

Solution:



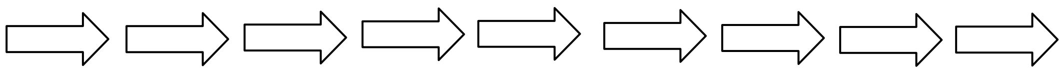
Suppose, for convenience, that the square was 1 x 1. (To get the answer to the stated question, multi[ly all results by 4). Put a little horizontal piece dead center in the "X". It is easy to compute the length of this configuration, as a function of the distance x, is: $F(x) = 4\sqrt{\frac{1}{4} + x^2} + 1 - 2x$

From the graph we see that one can get slightly under 2.75, as desired. (Note: it is possible to show that the minimum is exactly √3 + 1.

Problem of the Week

April 1, 2008

Reversing Arrows



You have 9 arrows which, for the moment, are all right facing. Whenever you like, you may reverse the direction of any 4 of them. You may make as many moves as you like. In particular, any given arrow may be switched from right to left and back again any number of times. Is it possible for you to make all arrows face left?

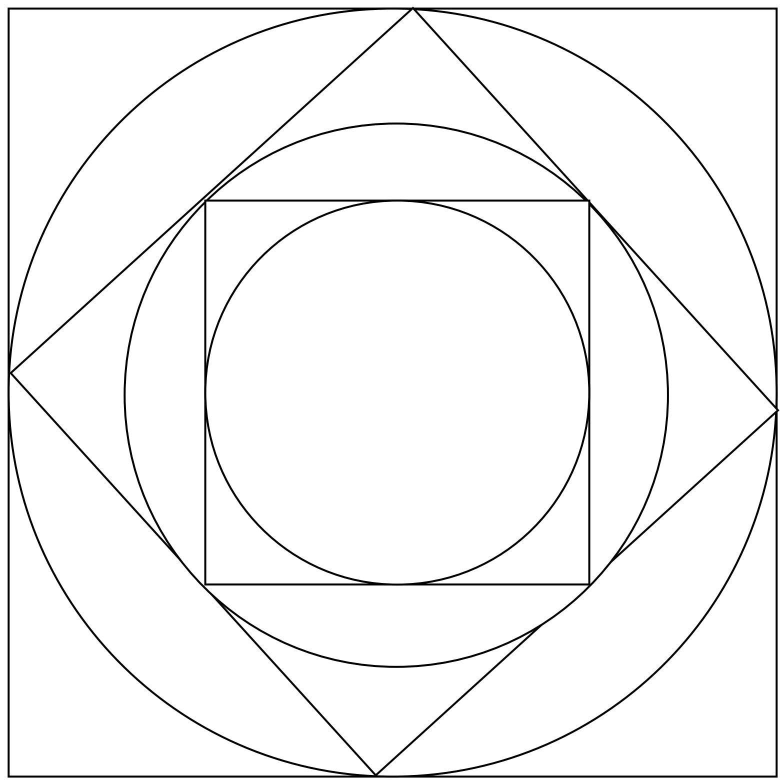
What if, instead of 4, a move consisted of reversing any 5 arrows?

Note: If you want to try to play this game, you might find it easier to do it with coins instead of arrows. Just start with 9 Heads, and turn over 4 at a time.

Problem of the Week

April 8, 2008

Scale



The figure above is made out of alternating squares and circles; you should imagine that the pattern goes on forever, though I stopped the drawing at some point. The largest square has side length 4. Is the first figure with area less than 1 a circle or a square? What about 1/10 ? 1/1,000 ?

Hint: What is the area of the second square?

Problem of the Week

April 8, 2008

A Problem of Scale

Scientists have found that some ants can lift more than ten times their body weight over their heads. Multiply by ten to find out how human weight lifters compare with the champions of the insect world.

--PBS website

Just how impressive is it that an ant can lift 10 (or 50) times its own weight? It sounds impressive. I can, maybe, lift my own weight. A professional can lift 5 times their own weight, maybe. So ants are stronger, right? But there is a big problem with this. Ants are a lot smaller than we are. Does that matter? How?

Strength, the strength of my arm, say, depends on a lot of things, but it doesn't really depend on the length of my arm. A long muscle is not going to be able to lift more than a short one. But the thickness does matter. Strength is proportional to the cross-sectional area of a muscle. *Weight,* on the other hand, is proportional to Volume.

Now try the ant problem. An ant is about ½ of an inch long. I am 6 feet tall. What scale factor gets me to ant size? Let's say I weigh 175 lbs and that I can lift 100 lbs. over my head.

1. How much would I weigh after being shrunk to ant size?

2. How much could I lift over my head after being shrunk?

3. How many multiples of my weight could I lift?

4. Who is stronger, me or the ant?

Conversely: A typical ant weighs something like 3 milligrams, or about .000007 pounds. Sticking with the same ½ inch size, and assuming the (tiny) ant could lift 50 times its own weight, how much would you expect it to be able to lift it were scaled up to my size? Could it lift its own weight?

Problem of the Week

April 22, 2008

The Dollar Auction Game

Here's a deceptively simple game introduced by the economist Martin Shubik in an attempt to capture some of thinking that goes into the arms race.

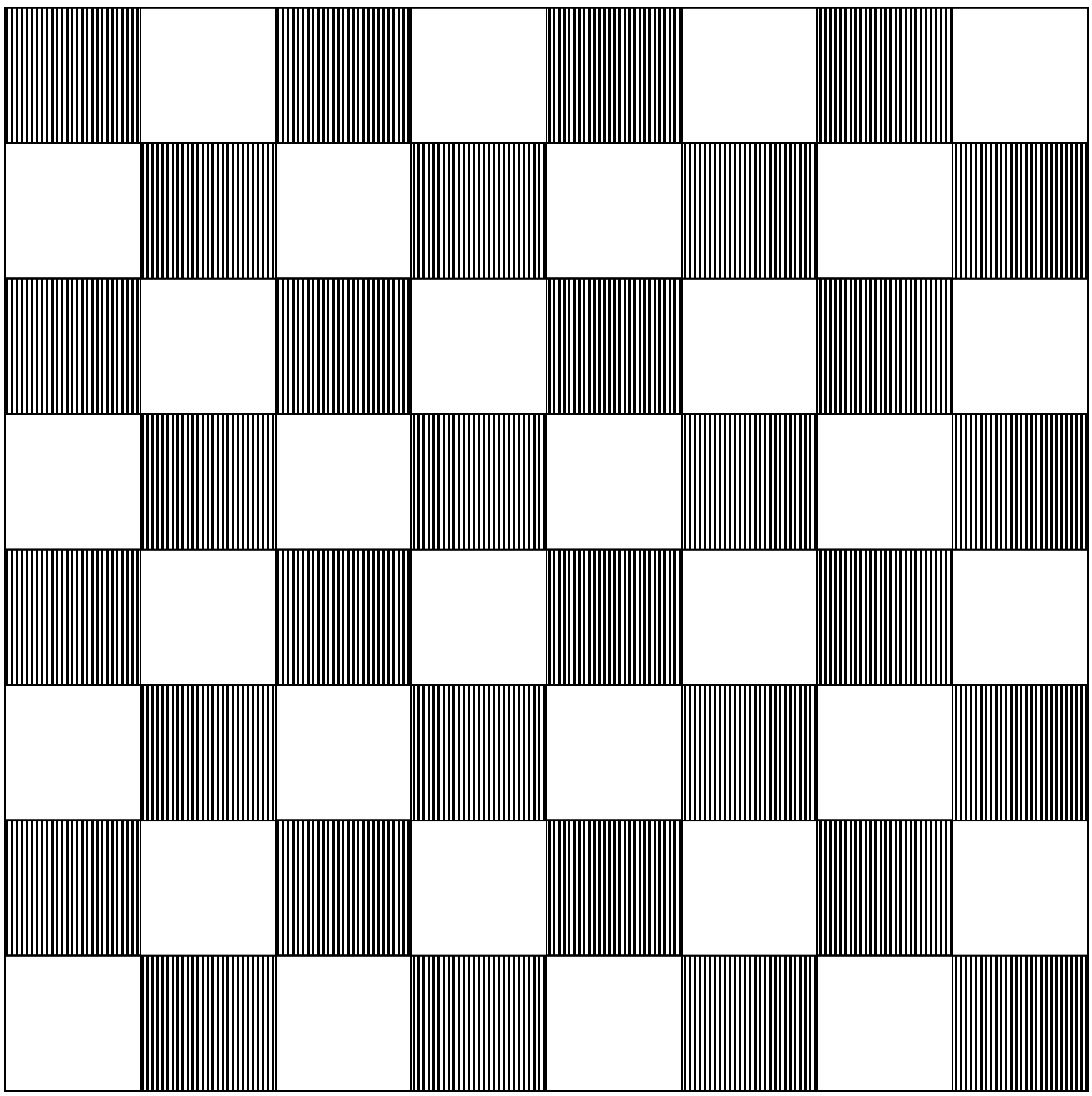
The game is played with an Auctioneer and 2 (or more) Bidders. The Auctioneer is auctioning off $1. The Bidders take turns in the usual way, with the dollar going to the highest Bidder (once the bidding has stopped). The High Bidder then pays the last amount bid and receives the dollar. The twist is that the Second Highest Bidder must also pay the last bid they made, but receives nothing for it.

Can you describe how the bidding in this game is likely to go?

Problem of the Week

April 29, 2007

The Queens Problem

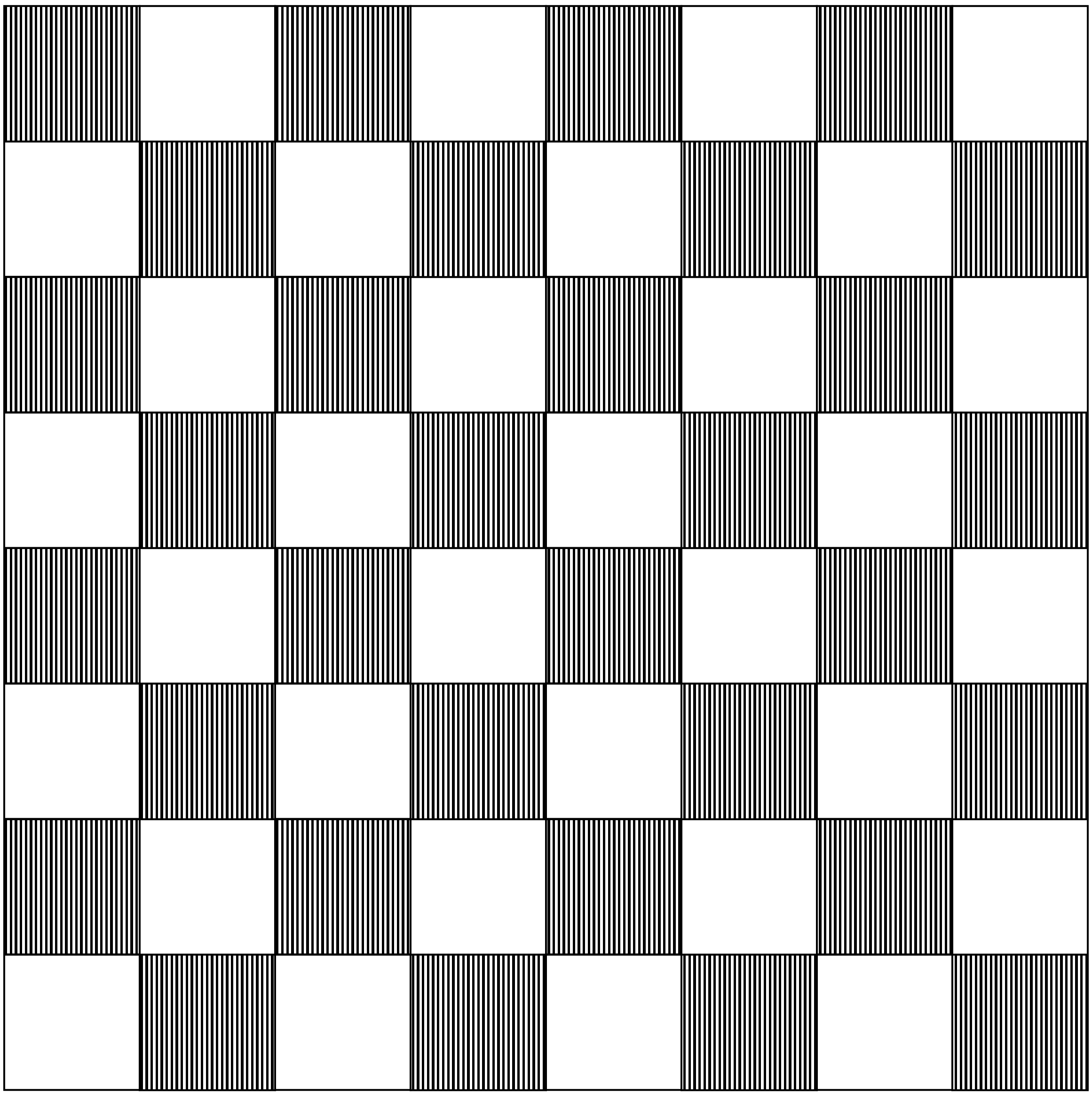


A classic problem: How many queens can you arrange on a regular 8 x 8 chessboard so that no two are attacking each other? The usual rules of chess apply (though there aren't white and black sides; each piece stands alone).

Problem of the Week

May 6, 2008

The Queens Problem II

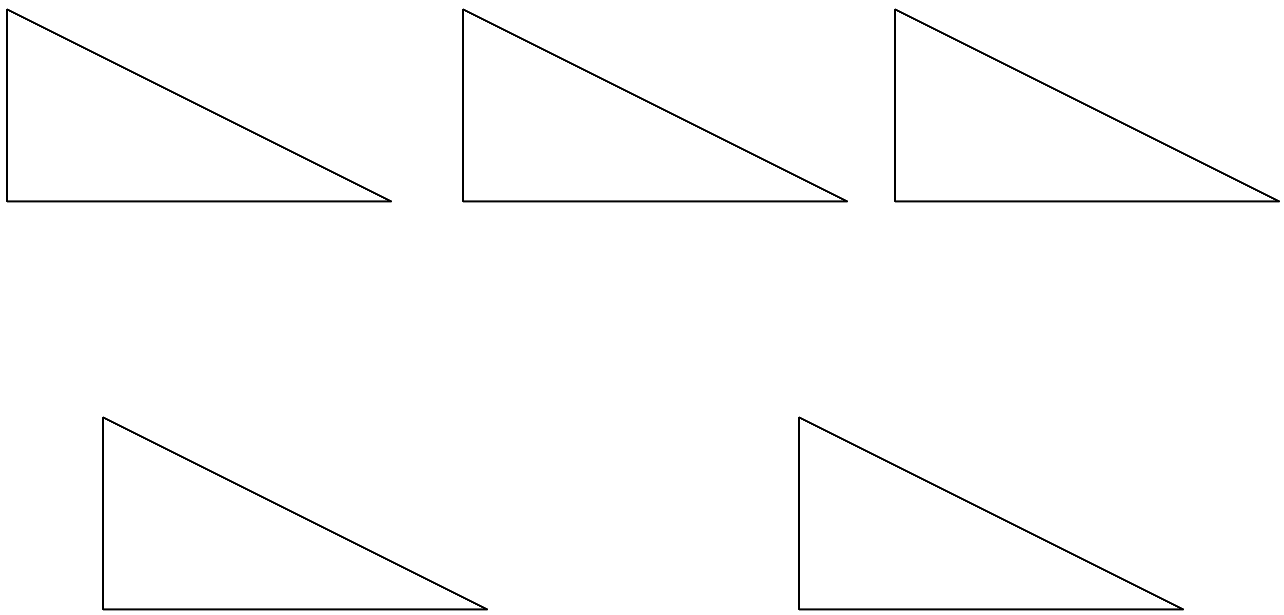


Now let's ask the opposite question: how few queens are needed so that every square on the board is under attack?

Problem of the Week

May 13, 2008

Here are five identical right triangles:

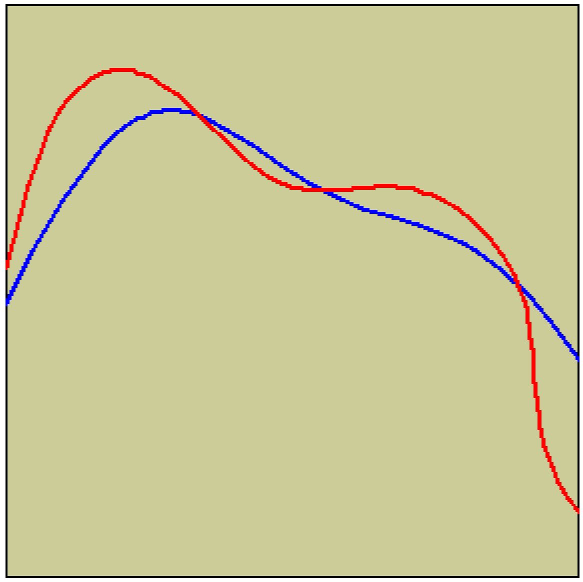


You may (and indeed should) cut these triangles out. Having done this, it is possible to cut one of them into two pieces…and then to rearrange the six pieces into a perfect square. Can you sort out how to do this?

Note: just to be clear, most cuts will not work. So far as I am aware, there is only one good way to cut up the sixth triangle.

Extra Question: This triangle is special; most right triangles will not work. Given that the height of one of these triangles is 1…how long is its base?

Problem of the Week May 20, 2008



In *The Adventure of the Priory School*, Sherlock Holmes is required to analyze the tracks left by a bicycle in a muddy field. He and Watson have the following conversation:

“This track, as you perceive, was made by a rider who was going from the direction of the school."

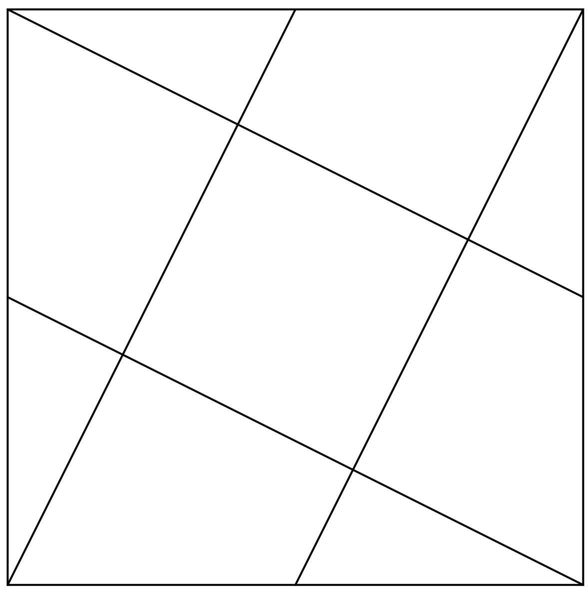
"Or towards it?"

"No, no, my dear Watson. The more deeply sunk impression is, of course, the hind wheel, upon which the weight rests. You perceive several places where it has passed across and obliterated the more shallow mark of the front one. It was undoubtedly heading away from the school.”

This poses a small problem. The argument Watson attributes to Holmes is surely insufficient: it is true one can tell which track was made by the rear wheel as Holmes suggests, but that knowledge alone doesn’t determine which way the bike was going. We may, of course, imagine that Watson omitted some details (the possibility that Holmes was in error being a circumstance which we shall not for a moment consider). Ah, but then we have to supply the details ourselves. So: given the tracks made by a single bike, how can you tell which way the bike was going? Try your method on the tracks shown above.

Problem of the Week

May 27, 2008



A square is cut up as you see it. Each slice begins at a corner of the square and ends at the midpoint of another side. It appears that these lines cut out a square from the middle. Is this so? Is this figure really a square? What is its area? (assume that each side of the original square had length 1).